B.SC. (MATH) - PART - II
Paper - IV

Topic: - Newton's Law of gravitation and planetary orbit

Q:- State and explain Newton's Law of gravitation

Statement:- Every particle in the universe attracts every other particle with the force directly proportional to the mass of each and inversely proportional to the ~~mass~~ square of their distance apart

Explanation of Newton's law of gravitation.

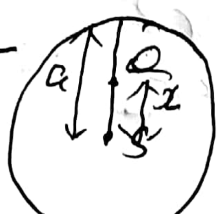
m_1 —— r —— m_2

Let m_1 and m_2 be the masses of two particles placed at at distance r apart. Then the mutual attraction is $\gamma \frac{m_1 m_2}{r^2}$ unit of force

where γ is constant, depending on the units of mass and length employed, and known as constant of gravitation.

Q!– If the planet is suddenly stopped in its orbit suppose it be circular then show that it would fall into the sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.

Soln:–



Let mass of sun $= M$

mass of planet $= m$

radius of circular orbit $= a$

By Newton's law of attraction on the mutual attraction $= \gamma \frac{Mm}{d^2}$

where d is the distance between sun and planet and γ is const.

The attraction of sun on the planet produces an acceleration

$$= \gamma \frac{M}{d^2}$$

The attraction of planet on the sun produces an acceleration

$$= \gamma \frac{m}{d^2}$$

But act in opposite directions

Thus the acceleration of planet relative to the sun

$$= \frac{\gamma(M+m)}{d^2} = \frac{\mu}{d^2} \quad [\mu = \gamma(M+m) \text{ (Say)}$$

Hence the acceleration of the planet relative to sun $= \frac{\mu}{d^2}$

Let $SQ = x$

Then the eqn. of motion is

$$\frac{d^2x}{dt^2} = -\frac{\omega}{x^2} \quad [x \text{ is decreasing}]$$

$$\Rightarrow v\frac{dv}{dx} = -\frac{\omega}{x^2}$$

$$\Rightarrow \int v\,dv = -\omega\int\frac{dx}{x^2}$$

$$\Rightarrow \frac{v^2}{2} = -\omega\left(-\frac{1}{x}\right) + C$$

$$\Rightarrow \frac{v^2}{2} = \frac{\omega}{x} + C$$

At the position of plament is P then $x = a$, $v = 0$ then by ① $C = -\frac{\omega}{a}$

So ① is becomes as

$$\frac{v^2}{2} = \frac{\omega}{x} - \frac{\omega}{a} = \omega\left(\frac{a-x}{ax}\right)$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 2\omega\left(\frac{a-x}{ax}\right) \quad \left[\because v = \frac{dx}{dt}\right]$$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{\frac{2\omega}{a}}\sqrt{\frac{a-x}{x}}$$

[$\because$ Motion is in the direction dx decreasing so negative sign taken]

$$\Rightarrow -\sqrt{\frac{2\omega}{a}}\int dt = \int\sqrt{\frac{x}{x-a}}\,dx$$

$$\Rightarrow -\sqrt{\frac{2\omega}{a}}\,t = \int\sqrt{\frac{a\cos^2\alpha}{a(1-\cos^2\alpha)}}\,(-2a)\cos\alpha\sin\alpha\,d\alpha$$

By putting $x = a\cos^2\alpha$

$\therefore dx = -2a\cos\alpha\sin\alpha\, d\alpha$

$\Rightarrow -\sqrt{\frac{2w}{a}}\, t = -a\int(1+\cos 2\alpha)\, d\alpha$

$\Rightarrow \sqrt{\frac{2w}{a}}\, t = a\left[\alpha + \frac{\sin 2\alpha}{2}\right] + c_1$

$\Rightarrow \sqrt{\frac{2w}{a}}\, t = a\left[\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\right] + c_1$ — (2)

In case of planet at P

$x = a$ and $t = 0$ then by (2), $c_1 = 0$

$\sqrt{\frac{2w}{a}}\, t = a\left[\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}}\right]$ — (3)

When the planet reaches at S then $t = \tau$, $x = 0$ then by (3)

$\sqrt{\frac{2w}{a}}\, \tau = a\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{2}a$ — (4)

If period of planet's revolution be T then we know that

$T = \frac{2\pi}{\sqrt{w}} a^{3/2}$

$\Rightarrow \sqrt{w}\, T = 2\pi a^{3/2}$ — (5)

By (4) and (5) we get

$\sqrt{\frac{2}{a}} \cdot \frac{\tau}{T} = \frac{a}{4a^{3/2}} = \frac{1}{4\sqrt{a}}$

$\therefore \tau = \frac{1}{4\sqrt{2}} T = \frac{\sqrt{2}}{8} T$

$\therefore$ Time $\tau = \frac{\sqrt{2}}{8} T$